Theorem: Suppose we have a power series with radius of convergence R. If $\sum c_n x^n$ converges and $|x| = R$, then the limit as a goes to R along the line arg(a)=arg(x) in the direction away from the origin of $\sum c_n a^n$ is $\sum c_n x^n$

Recommended level for proof: 6

Why do we want this: Because we know that taylor series work strictly inside the radius of convergence, but series like $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots = \ln(2)$ and $1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} - \cdots = \frac{\pi}{4}$ rely on knowing that the taylor series for $\ln(1+x)$ and $\arctan(x)$ respectively work at the boundary of their radius of convergence. If we can show that the power series that we get from using the log taylor series evaluated at 0.9, 0.99, 0.999, etc that we know are equal to ln(1.9), ln(1.99), ln(1.999) etc converge to $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$, which clearly converges itself, then it will mean that $1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots$ coincides with ln(2) (since ln is continuous there), so the taylor series will work at the edge of the radius of convergence whenever it actually converges there and the function we are expressing as a taylor series is continuous there, and we know already that it works inside the radius of convergence. So that is why we want this theorem.

Proof:

We will suppose that R=1 and that the point we are considering where the power series converges is the point x=1. This is because we can scale and rotate it as necessary afterwards, but this will simplify calculations.

Therefore we are supposing that $\sum_{n=0}^{\infty} c_n$ converges to a value we will call s and that the power series $\sum c_n x^n$ has radius of convergence 1.

Note that $c_n = \sum_{r=0}^{n} c_r - \sum_{r=0}^{n-1} c_r$. This seems like a complicated way of doing things but it will work out.

Now let x be a real number strictly between 0 and 1.

Now $\sum_{n=0}^{N} c_n x^n = c_0 + \sum_{n=1}^{N} c_n x^n = c_0 + \sum_{n=1}^{N}(\sum_{r=0}^{n} c_r - \sum_{r=0}^{n-1} c_r)x^n$. Now we can do a little trick: We hav the following terms in our sum:

$\sum_{r=0}^{1} c_r\, x$	$-\sum_{r=0}^{0} c_r\, x$
$\sum_{r=0}^{2} c_r\, x^2$	$-\sum_{r=0}^{1} c_r\, x^2$
$\sum_{r=0}^{3} c_r\, x^3$	$-\sum_{r=0}^{2} c_r\, x^3$
…	…
$\sum_{r=0}^{N} c_r\, x^N$	$-\sum_{r=0}^{N-1} c_r\, x^{N-1}$

Now we will separate out the top-right and bottom-left terms, and simplify the top right term to $c_0 x$. Then the sum of the rest of the terms (add each term to the one directly to the bottom-right of it) is exactly $\sum_{n=1}^{N-1} \sum_{r=0}^{n} c_r\,(x^n - x^{n+1})$. So $\sum_{n=0}^{N} c_n x^n = c_0 - c_0 x + \sum_{r=0}^{N} c_r\, x^N + \sum_{n=1}^{N-1} \sum_{r=0}^{n} c_r\,(x^n - x^{n+1})$. Therefore $\sum_{n=0}^{N} c_n x^n = c_0(1-x) + \sum_{r=0}^{N} c_r\, x^N + \sum_{n=1}^{N-1} \sum_{r=0}^{n} c_r\,(1-x)x^n$. But then we can put the term

$c_0(1-x)$ into the sum on the right, ie we get $\sum_{n=0}^{N} c_n x^n = \sum_{r=0}^{N} c_r\, x^N + (1-x)\sum_{n=0}^{N-1}\sum_{r=0}^{n} c_r\, x^n$. By the hypothesis of the theorem, the right hand side converges. Since $\sum_{r=0}^{N} c_r$ converges, it is bounded. Since x^N goes to 0 as N gets large, $\sum_{r=0}^{N} c_r\, x^N$ goes to 0 since $\sum_{r=0}^{N} c_r$ is bounded. Therefore when we take the limit as N goes to infinity that term vanishes and we get $\sum_{n=0}^{\infty} c_n x^n = (1-x)\sum_{n=0}^{\infty}\sum_{r=0}^{n} c_r\, x^n$. The sum on the right converges since differs from a convergent sum by something that approaches 0. Now our goal is to show that as x goes to 1 from below, $\sum_{n=0}^{\infty} c_n x^n$ approaches s. Note that since x<1, we can safely say that $(1-x)\sum_{n=0}^{\infty} x^n = 1$ (geometric series or generalized binomial theorem). Therefore we can subtract s from both sides of the equation we deduced above to get that $\sum_{n=0}^{\infty} c_n x^n - s = (1-x)\sum_{n=0}^{\infty}(\sum_{r=0}^{n} c_r - s)x^n$. Therefore if we can show that the right hand side tends to 0, we will know that the left hand side tends to 0 so we will be done. Now lets pick a number ε as small as we like then pick an M such that for all $n \geq M$, $|\sum_{r=0}^{n} c_r - s| < \frac{\varepsilon}{2}$, possible by definition of summing to infinity. Then we will write $\sum_{n=0}^{\infty}(\sum_{r=0}^{n} c_r - s)x^n$ as $\sum_{n=0}^{M-1}(\sum_{r=0}^{n} c_r - s)x^n + \sum_{n=M}^{\infty}(\sum_{r=0}^{n} c_r - s)x^n$.

$$\sum_{n=0}^{\infty} c_n x^n - s = (1-x)\sum_{n=0}^{M-1}\left(\sum_{r=0}^{n} c_r - s\right)x^n + \sum_{n=M}^{\infty}\left(\sum_{r=0}^{n} c_r - s\right)x^n$$

So by the triangle inequality

$$\left|\sum_{n=0}^{\infty} c_n x^n - s\right| \leq |1-x|\sum_{n=0}^{M-1}\left|\sum_{r=0}^{n} c_r - s\right| |x|^n + \sum_{n=M}^{\infty}\left|\sum_{r=0}^{n} c_r - s\right| |x|^n$$

$$\left|\sum_{n=0}^{\infty} c_n x^n - s\right| \leq |1-x|\sum_{n=0}^{M-1}\left|\sum_{r=0}^{n} c_r - s\right| |x|^n + \sum_{n=M}^{\infty}\frac{\varepsilon}{2}|x|^n$$

$$\left|\sum_{n=0}^{\infty} c_n x^n - s\right| \leq |1-x|\sum_{n=0}^{M-1}\left|\sum_{r=0}^{n} c_r - s\right| |x|^n + |1-x|\frac{\varepsilon}{2}\frac{|x|^M}{1-|x|}$$

(last line by geometric series). Since $0 < x < 1$, $|x|^M < 1$, so the last term is ast most $\frac{\varepsilon}{2}$. Since M does not depend on x and x is a positive real number less than 1, the term $|1-x|\sum_{n=0}^{M-1}|\sum_{r=0}^{n} c_r - s||x|^n$ is at most $|1-x|\sum_{n=0}^{M-1}|\sum_{r=0}^{n} c_r - s|$, which is just $|1-x|C$ for some constant C. Therefore this tends to 0 as x gets close enough to 1, in particular it is eventually less than $\frac{\varepsilon}{2}$. Therefore, for x close enough to 1,

$$\left|\sum_{n=0}^{\infty} c_n x^n - s\right| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon$$

This means the power series can be made arbitrarily close to s by making x sufficiently close to 1 from the left of 1 on the number line. This completes the proof of the theorem.